THE HOWE DUALITY AND THE PROJECTIVE REPRESENTATIONS OF SYMMETRIC GROUPS

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Abstract. The symmetric group $\mathfrak{S}_n$ possesses a nontrivial central extension, whose irreducible representations, different from the irreducible representations of $\mathfrak{S}_n$ itself, coincide with the irreducible representations of the algebra $\mathfrak{A}_n$ generatored by indeterminates $\tau_{i,j}$ for $i \neq j$, $1 \leq i, j \leq n$ subject to the relations

$$\tau_{i,j} = -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{k,l} = -\tau_{k,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{k,l\} = \emptyset;$$
$$\tau_{i,j}\tau_{j,k}\tau_{i,j} = \tau_{j,k}\tau_{i,j}\tau_{j,k} = -\tau_{i,k} \quad \text{for any } i,j,k,l.$$

Recently M. Nazarov realized irreducible representations of $\mathfrak{A}_n$ and Young symmetrizers by means of the Howe duality between the Lie superalgebra $\mathfrak{q}(n)$ and the Hecke algebra $H_n = \mathfrak{S}_n \circ C_n$, the semidirect product of $\mathfrak{S}_n$ with the Clifford algebra C_n on n indeterminates.

Here I construct one more analog of Young symmetrizers in H_n as well as the analogs of Specht modules for $\mathfrak{A}_n$ and H_n.

§1. Introduction

Lately we witness an increase of interest in the study of representations of symmetric groups. In particular, in their projective representations.

Recall that the symmetric group $\mathfrak{S}_n$ has a nontrivial central extension whose irreducible representations do not reduce to those of $\mathfrak{S}_n$ but coincide (may be identified) with the irreducible representations of the algebra $\mathfrak{A}_n$ determined by generators $\tau_{i,j}$ for $i \neq j$, $1 \leq i, j \leq n$ subject to the relations

$$\begin{aligned} &\tau_{i,j} = -\tau_{j,i}, \quad \tau_{i,j}^2 = 1, \quad \tau_{i,j}\tau_{k,l} = -\tau_{k,l}\tau_{i,j} \text{ if } \{i,j\} \cap \{k,l\} = \emptyset; \\ &\tau_{i,j}\tau_{j,k}\tau_{i,j} = \tau_{j,k}\tau_{i,j}\tau_{j,k} = -\tau_{i,k} \quad \text{for any } i,j,k,l. \end{aligned} \tag{1.1}$$

In [N1] Nazarov realized irreducible representations of $\mathfrak{A}_n$ by means of an orthogonal basis constructed in each of the spaces of the representations and indicating the action of the generators $\tau_{i,i+1}$ on them (an analog of the Young orthogonal form). In [N2], with the help of an "odd" analog of the degenerate affine Hecke algebra Nazarov constructed elements of the algebra $H_n = \mathfrak{S}_n \circ C_n$, the semidirect product of $\mathfrak{S}_n$ with the Clifford algebra C_n on n indeterminates. The elements of H_n serve as analogs of Young symmetrizers.

Here I construct one more analog of Young symmetrizers in H_n as well as analogs of Specht modules (cf. [Ja]) for the algebras $\mathfrak{A}_n$ and H_n. This construction is based on another form of expression of Young symmetrizers for $\mathfrak{S}_n$.

Namely, let t be a Young tableau (i.e., a Young diagram filled in with numbers 1 to n), R_t and C_t the row and column stabilizers of t;

$$\rho_t = \sum_{\sigma \in R_t} \sigma, \quad \kappa_t = \sum_{\sigma \in C_t} \varepsilon(\sigma)\sigma.$$

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Then, up to a constant factor, the Young symmetrizer of t is

$$e_t = \kappa_t \rho_t. \tag{1.2}$$

Let us represent the Young symmetrizer differently. Let I be the sequence obtained by reading the tableau t along columns left to right and downwards. For each $i \in I$ the *Jucys–Murphy elements* ([J], [M]) are defined to be

$$x_i = \sum_{\alpha \text{ preceeds } i} s_{\alpha i}, \text{ where the } s_{\alpha i} \in \mathfrak{S}_n \text{ are transpositions.}$$

One can verify that the x_i commute with each other. Set

$$\tilde{\kappa}_t = \prod_{i \in I} (j - x_i), \quad \text{where } j \text{ is the number of the column occupied by } i.$$

It is subject to a direct verification that e_t can be also expressed as

$$e_t = \tilde{\kappa}_t \rho_t. \tag{1.3}$$

I use this other representation of the Young symmetrizers to construct the corresponding elements in H_n and prove that theyse corresponding elements are idempotents that generate isotypical ideals. With the help of these elements and their analogs for $\mathfrak{A}_n$ I construct a realization of multiples of irreducible modules similar to Specht modules, cf. [Ja].

The proofs are based on the Howe duality between Lie superalgebra $\mathfrak{q}(n)$ and H_n.

§2. Background

Let $\mathfrak{S}_n$ be the symmetric group, C_n the Clifford algebra generated by n indeterminates $p_1, \dots, p_n$ subject to the relations

$$p_i^2 = -1, \; p_i p_j + p_j p_i = 0 \text{ for } i \neq j.$$

The symmetric group acts on C_n permuting the generators, so we can form a semidirect product $H_n = \mathfrak{S}_n \circ C_n$. Set

$$\tau_{i,j} = \frac{1}{\sqrt{2}} (p_i - p_j) s_{i,j}.$$

As is not difficult to verify, the relations (1.1) hold; hence, the algebra generated by the $\tau_{i,j}$ is isomorphic to $\mathfrak{A}_n$. Besides, $\mathfrak{A}_n$ supercommutes with C_n; hence, $H_n = \mathfrak{A}_n \otimes C_n$, as *super*algebras if we define parity in $\mathfrak{A}_n$ by setting $p(\tau_{i,j}) = \bar{1}$.

Let V be a superspace of superdimension (n, n) with the fixed basis $\{e_i\}_{i=1}^{n} \cup \{e_{\bar{i}}\}_{\bar{i}=\bar{1}}^{\bar{n}}$ and the odd operator

$$Q : e_i \mapsto (-1)^{\bar{i}} e_{\bar{i}}; \quad e_{\bar{i}} \mapsto e_i.$$

The (super)centrilizer of Q in $\mathrm{Mat}(V)$ is denoted by $\mathrm{Q}(V)$, cf. [BL]. We denote the Lie superalgebras associated with the associative superalgebras $\mathrm{Mat}(V)$ and $\mathrm{Q}(V)$ by $\mathfrak{gl}(V)$ and $\mathfrak{q}(V)$, respectively.

As is shown in [S1], the algebras $\mathfrak{q}(V)$ and H_k constitute a Howe-dual pair in the superspace $W = V^{\otimes k}$. Therefore, the following decomposition takes place:

$$W = \bigoplus_{\lambda} 2^{-\delta(\lambda)} T^\lambda \otimes V^\lambda, \tag{2.1}$$

where λ runs over strict partitions of k, T^λ is an irreducible (in supersence) H_k-module, V^λ an irrreducible $\mathfrak{q}(n)$-module and $\lambda_{n+1} = 0$ and where

$$\delta(\lambda) = \begin{cases} 0 & \text{if the number of nonzero parts of } \lambda \text{ is even} \\ 1 & \text{otherwise.} \end{cases}$$

Select a basis in $\mathfrak{q}(n)$: let the e_i^* be the left dual basis to the $\{e_i\}$, $i = 1, \dots n;\ \bar{1} \dots \bar{n}$; set

$$E_{i,j} = e_i \otimes e_j^* + e_{\bar{i}} \otimes e_{\bar{j}}^*;\ F_{i,j} = e_i \otimes e_{\bar{j}}^* + e_{\bar{i}} \otimes e_j^*.$$

Then $\mathfrak{h} = Span(E_{i,i}$ and $F_{i,i} : i = 1, \dots, n)$ is a Cartan subalgebra in $\mathfrak{q}(n)$ and $\mathfrak{b} = Span(E_{i,j}$ and $F_{i,j} : i \leq j)$ is a Borel subalgebra, cf. [Pe].

If λ is a strict partition and $\lambda_{n+1} = 0$, then λ can be interpreted as a linear functional on $\mathfrak{h}_{\bar{0}}$: set

$$\lambda(E_{ii}) = \lambda_i. \tag{2.2}$$

Let R^λ be the H_k-module equal to the direct sum of $2^{(l(\lambda)-\delta(\lambda))/2}$ copies of T^λ. It is not difficult to see that R^λ coincides with the set of $\mathfrak{b}$-highest weight vectors of weight λ in W.

2.1. Lemma . *Let μ be a strict partition, $\mu_{n+1} = 0$. Let V^μ be an isotypical modul of type μ and n_μ the multiplicity of the highsest weight vector in V^μ. Then the highest weights of $V^\mu \otimes V$ are of the form $\mu + \varepsilon_i$, where the ε_i are the weights of V, and their multiplicity is equal to $2n_\mu$.*

Proof follows easily from the multiplication table of the projective Schur functions, see [P]. □

2.2. Lemma . *Let $\mathfrak{g} = \mathfrak{q}(n)$; $\mathfrak{b}$ and $\mathfrak{h}$ be defined as above and V a $\mathfrak{g}$-module. Let V_λ^+ be the set of $\mathfrak{b}$-highest vector of weight λ.*

If $u \in U(\mathfrak{g})$ and $uV_\lambda^+ \subset V_\lambda^+$, then there exists $w \in U(\mathfrak{h})$ such that $u|_{V_\lambda^+} = w|_{V_\lambda^+}$.

Proof. Let $u = \sum u_\alpha$ be the weight decomposition of $u \in U(\mathfrak{g})$ with respect to $\mathfrak{h}_{\bar{0}}$, the even part of the Cartan subalgebra $\mathfrak{h}$. Therefore, if $v \in V_\lambda$, then $uv = \sum_\alpha u_\alpha v$ and, if $u_\alpha v \neq 0$, then the weight of $u_\alpha v$ is equal to $\lambda + \alpha$. Thus, we may assume that $u = u_0 \in U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$, where $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$ is the centralizer of $\mathfrak{h}$.

Thanks to [S2], we know that $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}} \cong U(\mathfrak{h}) \oplus L$, where $L = U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}} \cap U(\mathfrak{g})\mathfrak{b}^+$, where $\mathfrak{b}^+$ is the linear span of the positive roots in $\mathfrak{b}$, is a twosided ideal in $U(\mathfrak{g})^{\mathfrak{h}_{\bar{0}}}$.

Hence, $u = w + u_1$, where $w \in U(\mathfrak{h})$ and $u_1 \in L$; this implies that $uv = wv$ for $v \in V_\lambda$. □

§3. Specht modules over H_k

Let λ be a strict partition and t the shifted tableau of the form λ, where $\lambda_1 > \lambda_2 > \dots > \lambda_n > 0$ and $\sum \lambda_i = k$. Let us fill in the tableau with the numbers 1 to k and define the functional λ on the Cartan subalgebra $\mathfrak{h}$ by setting

$$\lambda(E_{ii}) = \lambda_i; \quad \lambda(F_{ii}) = 0.$$

In $W = V^{\otimes k}$, where V is the standard $\mathfrak{q}(V)$-module of dimension (n, n), consider the submodule M^λ consisting of the vectors of weight λ.

Again, let R_t be the row stabilizer of t and $\rho_t = \sum_{\sigma \in R_t} \sigma$. Let I be the sequence obtained by reading the tableau t downwards and from left to right. For $i \in I$ define: $\pi_i = \sum \tau_{\alpha,i}$, where the sum runs over all the α's, $\alpha \in I$, that preceed $i \in I$.

It is subject to a direct verification that

$$\pi_i \pi_j + \pi_j \pi_i = 0 \text{ for } i \neq j.$$

These are odd analogs of the Jucys-Murphy elements. Set

$$\kappa_t = \prod_{i \in I} (\frac{1}{2} j(j+1) - \pi_i^2),$$

where j is the number of the column occupied by i.

3.1. Theorem . *Let $v_t \in M^\lambda$ be the vector whose stabilizer is R_t. Then $\kappa_t(v_t) \neq 0$, $\kappa_t(v_t) \in R^\lambda$ and $\kappa_t(M^\lambda) = C_k\kappa_t(v_t)$.*

Proof. Induction on $\sum \lambda_i = k$. It suffices to assume that t is filled in consequently columnwise, from left to right and downwards. Let s be the tableau obtained from t by deleting the last cell and μ the corresponding partition while l is the length of the last column and r is its number. Then $\kappa_t = \kappa_s \cdot \kappa_k$, where κ_s corresponds to tableau s and $\kappa_k = \frac{1}{2}r(r+1) - \pi_k^2$.

By induction, $\kappa_s(M^\mu) = C_{k-1}\kappa_s(v_s) \subset R^\mu$. Therefore, $\kappa_s(M^\nu) = 0$ for any $\nu > \mu$ ordered with respect to *dominance*, cf. [M]. Hence,

$$\kappa_s(M^\lambda) = \bigoplus_{i=1}^{n} \kappa_s(M^{\lambda-\varepsilon_i}) \otimes e_i.$$

If $i > l$, then $\lambda - \varepsilon_i > \lambda - \varepsilon_l = \mu$ and, by induction, $\kappa_s(M^{\lambda-\varepsilon_i}) = 0$. Hence,

$$\kappa_s(M^\lambda) = \bigoplus_{i=1}^{l} \kappa_s(M^{\lambda-\varepsilon_i}) \otimes e_i.$$

The inequality $\nu \geq \lambda - \varepsilon_i$ true for $i < l$ implies that $\nu \geq \mu$, so the same irreducible H_{k-1}-modules enter the decomposition of $M^{\lambda-\varepsilon_i}$ as those that enter that of M^μ. Therefore, if $m_i \in M^{\lambda-\varepsilon_i}$ and $\kappa_s(m_i) \neq 0$, then there exist $m \in M^\mu$ and a homomorphism $\varphi_i : M^\mu \longrightarrow M^{\lambda-\varepsilon_i}$ such that

$$\varphi_i(m) = m_i.$$

Applying κ_s to this identity we get

$$\kappa_s(m_i) = \kappa_s(\varphi_i(m)) = \varphi_i(\kappa_s(m)) \in C_{k-1}\kappa_s(v_s).$$

The Howe duality between $\mathfrak{q}(n)$ and H_{k-1} allows us to assume that $\varphi_i \in U(\mathfrak{q}(n))$. This proves that $\kappa_s(M^\lambda) \subset V^\mu \otimes V$, where V^μ is the $\mathfrak{q}(n)$-submodule of $V^{\otimes(k-1)}$ generated by $\kappa_s(M^\mu)$. By Lemma 2.1 the highest weights of $V^\mu \otimes V$ are of the form $\mu + \varepsilon_i$, so the possible weights are only

$$\mu + \varepsilon_1,\ \mu + \varepsilon_l = \lambda \text{ and } \mu + \varepsilon_i < \mu + \varepsilon_l \text{ if } i > l.$$

So the submodule generated by the weights $\mu + \varepsilon_i$ for $i > l$ does not contain weight λ; hence, neither does it contain $\kappa_s(M^\lambda)$.

Therefore, $\kappa_s(M^\lambda)$ is contained in the submodule generated by the highest weight vectors of weight $\mu + \varepsilon_1$ and $\mu + \varepsilon_l$. By Lemma 2.1 all the highest weight vectors form a free C_k-module with two generators whose weights are $\mu + \varepsilon_1$ and $\mu + \varepsilon_l$; so in each of these submodules the operator π_k^2 acts by multiplying by a constant. Let $v \in V^{\otimes(k-1)}$ and $e_i \in V$. Then it is not difficult to verify that

$$\pi_k(v \otimes e_i) = \frac{1}{\sqrt{2}} \sum_{1 \leq i \leq n} (F_{ij} - p_k E_{ij}(v \otimes e_i)),$$

where p_i is the change of parity operator in the i-th factor of $V^{\otimes k}$.

If v is a highest weight vector of weight μ, than $v \otimes e_1$ is a highest weight vector of weight $\mu + \varepsilon_1$ and

$$\pi_k(v \otimes e_1) = \frac{1}{\sqrt{2}}(F_{11} - p_k E_{11})(v \otimes e_1),$$

hence,

$$\begin{aligned} &\pi_k^2(v \otimes e_1) = -\tfrac{1}{2}(F_{11} - p_k E_{11})^2(v \otimes e_1) = \\ &\tfrac{1}{2}(E_{11}^2 - E_{11})^2(v \otimes e_1) = \\ &\tfrac{1}{2}((r+1)^2 - (r+1))(v \otimes e_1) = \tfrac{1}{2}r(r+1)(v \otimes e_1). \end{aligned}$$

Therefore, it suffices to demonstrate that

$$\kappa_t(v_t) = \kappa_t(v_s \otimes e_l) = \kappa_k(\kappa_s(v_s \otimes e_l) \neq 0.$$

It is not difficult to verify that if v is the highest weight vector, then

$$\pi_k^2(v \otimes e_l) = -\frac{1}{2}\sum_{k \leq i}[(E_{ik}^{(2)}v) \otimes e_k + p_k(F_{ik}^{(2)}v) \otimes e_k] + 2(\sum E_{kk}v) \otimes e_i,$$

where

$$E_{ik}^{(2)} = \sum_{k \leq j \leq i}(E_{ij}E_{jk} - F_{ij}F_{jk}),\ F_{ik}^{(2)} = \sum_{k \leq j \leq i}(E_{ij}F_{jk} - F_{ij}E_{jk}).$$

Therefore,

$$\pi_k^2(v \otimes e_l) = \frac{1}{2}\left(E_{ll}^{(2)} - E_{ll} + 2(E_{11} + \cdots + E_{ll})\right) v \otimes e_k + \dots,$$

where $\dots$ replace the sum of terms of the form $u \otimes e_i$ with $i < l$ and

$$\begin{aligned}&\pi_k^2(v \otimes e_l) = \tfrac{1}{2}\left(\lambda_l^2 - \lambda_l + 2(\lambda_1 + \cdots + \lambda_l)\right) v \otimes e_l + \cdots = \\ &\tfrac{1}{2}\left((r-l)(r-l-1) + 2(r + \cdots + r - (l-1) + 1 + r - l)\right) v \otimes e_l + \cdots = \\ &\tfrac{1}{2}(r^2 - r + 2l - 2)v \otimes e_l + \dots\end{aligned}$$

Consequently,

$$\kappa_k(v \otimes e_l) = \frac{1}{2}(r^2 + r - r^2 + r - 2l + 2)(v \otimes e_l) + \cdots = (r - l + 1)(v \otimes e_l) + \cdots \neq 0.$$

□

3.2. *Remark* . By [N2] (Th. 7.2) for any strict partition μ there exist elements $\psi_S \in R^\mu$, where S is the standard tableau of type μ, such that $p\psi_S$ constitute a basis of R^μ while p runs over C_n. Moreover, the action of the elements

$$x_k = \sum_{i<k}(s_{ik} + s_{ik}p_ip_k)$$

is given by the formula

$$x_k\psi_S = C_S(k)\psi_\Lambda,$$

where $C_S(k) = \sqrt{(j-i)(j-i+1)}$ and k occupies the (i, j)-th slot.

In the above notations

$$x_k = \sqrt{2}p_k\pi_k,$$

hence, $x_k^2 = 2\pi_k$ and

$$\begin{aligned}&\kappa_t = \prod_i \tfrac{1}{2}\left(j(j+1) - \pi_i^2\right) = \\ &\prod_i \tfrac{1}{2}\left(j(j+1) - x_i^2\right) = 2^{-k}\prod_i\left(j(j+1) - x_i^2\right).\end{aligned}$$

As earlier, we assume that the tableau is filled in left to right and downwards along columns.

Let us show that if $\mu \geq \lambda$, then $\kappa_t\psi_S = 0$ for any standard tableau S of type μ and distinct from t.

Let k be the number occupying the last place of the last column in the tableau t (filled in left to right and downwards). Let t^* be the tableau obtained from t by deleting the box with k; let S^* be similarly constructed from S. If $\mu^* \geq \lambda^*$ (shapes of S^* and t^*, respectively), then the induction hypothsis applies. If $\mu^* < \lambda^*$, then $\mu_1 = \lambda_1 + 1$ and k occupies the last

position of the first row of S. Hence, if j is the number of the column of t occupied by k, then

$$\pi_k^2(v_S) = \frac{1}{2}(j+1-1)(j+1-1+1)v_S = \frac{1}{2}j(j+1)v_S.$$

Hence, $\left(\frac{1}{2}j(j+1) - \pi_k^2\right)\psi_S = 0$ and $\kappa_t\psi_S = 0$.

This argument shows that κ_t is the projection in M^Λ onto the subspace generated by ψ_t.

At the same time

$$\kappa_k\psi_t = \frac{1}{2}\left(j(j+1) - (j-i)(j-i+1)\right)\psi_t = \left(ij - \frac{1}{2}i(i-1)\right)\psi_t \neq 0.$$

3.3. Specht modules. Recall that the *Specht module* for a strict partition λ is the submodule in M^λ generated by the vectors $\kappa_t v_t$ for all tableaux t, where v_t is the vector whose stabilizer is equal to R_t.

3.3.1. Theorem . *Specht module is equal to R^λ. It is isotypical and its H_k-centralizer is isomorphic to the Clifford algebra with $l(\lambda)$ generators.*

Proof. Let us realize R^λ as the set of highest weight vectors in $V^{\otimes k}$. By the Howe duality between $U(\mathfrak{q}(V))$ and H_k the algebra of H_k-homomorphisms is generated by $U(\mathfrak{q}(V))$. But thanks to Lemma 2.2 we may assume that the algebra of H_k-homomorphisms is generated by $U(\mathfrak{h})$ for the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{q}(V)$. If $\lambda_i = 0$, then in our notations for the basis of $\mathfrak{h}$ we have

$$F_{ii}^2 = E_{ii} = \lambda_i = 0 \text{ in } R^\lambda$$

and, therefore, $\ker F_{ii}$ is an $\mathfrak{h}$-submodule in the set $(V^\lambda)^+$ of highest weight vectors. But $(V^\lambda)^+$ is irreducible as $\mathfrak{h}$-module; hence, $F_{ii}|_{(V^\lambda)^+} = 0$ which implies $F_{ii}|_{R^\lambda} = 0$. Thus, the algebra of H_k-homomorphisms is generated by the F_{ii} for i such that $\lambda_i \neq 0$.

By Theorem 3.1 the Specht module is contained in R^λ and any homomorphism of M^λ sends R^λ into itself. Hence, the Specht module coinsides with R^λ. □

3.3.2. Corollary . *Set $p_t^i = \sum p_\alpha$, where the α belongs to the i-th column of tableau t. Then for any H_k-module endomorphism φ of M^λ we have*

$$\varphi(\kappa_t v_t) = f \cdot \kappa_t v_t,$$

where f belongs to the subalgebra of C_k generated by $p_t^1, \dots, p_t^{l(\lambda)}$.

Proof. Let us realize M^λ as the subset of vectors of weight λ in $V^{\otimes k}$. Then any endomorphism of M^λ may be identified with an element of $U(\mathfrak{q}(V))$; the restriction of this endomorphism on R^λ may be identified with an element of $U(\mathfrak{h})$.

By Theorem 3.3.1 the endomorphism algebra of R^λ is generated by the F_{ii} and to prove the corollary, it suffices to verify it for these elements. We have

$$F_{ii}(\kappa_t v_t) = \kappa_t(F_{ii}v_t) = \kappa_t(p_t^i v_t) = p_t^i\kappa_t(v_t)$$

□

3.3.3. Corollary . *Let $\varphi : M^\lambda \longrightarrow M^\lambda$ be an H_k-module endomorphism given by the formula $\varphi(v_t) = \rho_t\kappa_t(v_t)$. Then $\varphi|_{R^\lambda} = c \in \mathbb{C}$.*

Proof. Let us show that φ commutes with the endomorphisms F_{ii}. Indeed,

$$\begin{aligned}
&\varphi \cdot F_{ii}(v_t) = \varphi(p_t^i v_t) = p_t^i\varphi(v_t) = p_t^i\rho_t\kappa_t(v_t);\\
&F_{ii} \cdot \varphi(v_t) = F_{ii}\rho_t\kappa_t(v_t) = \rho_t\kappa_t(F_{ii}v_t) =\\
&\rho_t\kappa_t(p_t^i v_t) = \rho_t p_t^i\kappa_t(v_t) = p_t^i\rho_t\kappa_t(v_t).
\end{aligned}$$

The latter identity holds thanks to the fact that p_t^i commutes with ρ_t.

Thus, $\varphi \cdot F_{ii}(v_t) = F_{ii} \cdot \varphi(v_t)$ and, since the elements v_t generate the H_k-module M^λ, we have

$$\varphi \cdot F_{ii} = F_{ii} \cdot \varphi.$$

Since $\varphi(R^\lambda) \subset R^\lambda$ for any endomorphism φ of M^λ, it follows that $\varphi|_{R^\lambda}$ is an element from the centralizer of R^λ. But by Theorem 3.3.1 the centralizer is the Clifford algebra $C_{l(\lambda)}$. But φ is an even central element of $C_{l(\lambda)}$, hence, φ is a constant. □

3.3.4. Corollary . *Set $e_t = \kappa_t \rho_t$. Then*

$$e_t^2 = c \cdot e_t \text{ for } c \in \mathbb{C}, c \neq 0$$

and the algebra $e_t H_k e_t$ is isomorphic to $C_{l(\lambda)}$ and is generated by the p_t^i for $1 \leq i \leq l(\lambda)$.

Proof. Thanks to Corollary 3.3.3

$$\varphi(\kappa_t(v_t)) = \kappa_t \rho_t \kappa_t(v_t) = c\kappa_t(v_t)$$

or, equivalently,

$$e_t^2 = c \cdot e_t.$$

Since the constant term of $\kappa_t \rho_t$, equal to the coefficient of v_t in $\kappa_t(v_t)$, is nonzero, as follows from the proof of Theorem 3.3.1, then the routine arguments with the help of the bases (cf. [W]) shows that $c \neq 0$. □

Moreover, the algebra $e_t H_k e_t$ is *anti*-isomorphic to the algebra of H_k-endomorphism of the submodule of M^λ generated by $\kappa_t(v_t)$. Denote the latter module by R_1^λ. Thanks to Corollary 3.3.2 if φ is an endomorphism of R^λ, then $\varphi(R_1^\lambda) \subset R_1^\lambda$ and the restriction map $\varphi|_{\kappa_t(v_t)}$ determines an antiisomorphism of $\mathrm{End}_{H_k}(R^\lambda)$ into the algebra generated by the p_t^i for $1 \leq i \leq l(\lambda)$. But the latter algebra is isomorphic to $C_{(\lambda)}$; hence, $\varphi|_{\kappa_t(v_t)}$ is an anti-isomorphism and $R_1^\lambda = R^\lambda$.

§4. The Specht modules over $\mathfrak{A}_k$

In this section we construct analogs of the modules M^λ and the Specht modules R^λ for $\mathfrak{A}_k$.

First, we need the following statement.

4.1. Lemma . *Let $\pi_1 = \tau_{12}, \dots, \pi_2 = \tau_{13} + \tau_{23}, \dots, \pi_k = \sum_{\alpha<k} \tau_{\alpha k}$ be odd analogs of Jucys–Murphy's elements. Then*

$$e_k = \prod_{i \geq 2} \frac{2}{i(i-1)} \pi_i^2$$

is an idempotent and $e_k \mathfrak{A}_k e_k$ is isomorphic to the Clifford algebra with $k-1$ generators.

Proof. It is easy to verify by induction that

$$e_k = \frac{1}{k!} \sum_{\alpha \geq 0} (-1)^\alpha 2^{k-\alpha-1} \Sigma_{2\alpha+1},$$

where $\Sigma_{2\alpha+1}$ is the sum of all elements from $\mathfrak{A}_k$ of the form $\tau_{i_1 i_2} \tau_{i_2 i_3} \dots \tau_{i_{2\alpha} i_{2\alpha+1}}$. This implies that e_k is a central element that does not vary under the replacement of the sequence $\pi_i = \sum_{\alpha<i} \tau_{\alpha i}$ with $\pi_{\sigma(i)} = \sum_{\alpha<i} \tau_{\sigma(\alpha)\sigma(i)}$ for any i and $\sigma \in \mathfrak{S}_k *$. It is not difficult to verify that

$$(\tau_{12} + \tau_{23} + \tau_{31}) \cdot (2 - \tau_{12}\tau_{23} + \tau_{13}\tau_{32}) = 0.$$

Since

$$2-\tau_{12}\tau_{23}+\tau_{13}\tau_{32}=\pi_2^2=(\tau_{13}+\tau_{23})^2$$

is a factor in the expression for e_k, it follows that

$$(\tau_{12}+\tau_{23}+\tau_{31})e_k=0.$$

From symmetry considerations

$$(\tau_{ij}+\tau_{jl}+\tau_{li})e_k=0 \text{ for any distinct } i,j,l\in\{1,\dots k\}.$$

Further on,

$$\begin{array}{l}\pi_i^2=(\sum\tau_{\alpha i})^2=i-1-\sum(\tau_{\alpha\beta}\tau_{\beta i}+\tau_{\beta\alpha}\tau_{\alpha i})=\\ i-1-(\sum(1+\tau_{\alpha\beta}\tau_{\beta i}+\tau_{\beta\alpha}\tau_{\alpha i})-1)=\\ i-1+\frac{1}{2}(i-1)(i-2)-\sum\tau_{\alpha\beta}(\tau_{\alpha\beta}+\tau_{\beta i}+\tau_{i\alpha}).\end{array}$$

Hence, $\pi_i^2e_k=\frac{1}{2}(i-1)(i-2)e_k$ and, therefore,

$$e_k^2=\left(\prod_{i\geq 2}\frac{2}{i(i-1)}\pi_i^2\right)e_k=e_k.$$

Furthermore, since e_k is a central element, then $e_k\mathfrak{A}_ke_k=\mathfrak{A}_ke_k$. Let I be the ideal in $\mathfrak{A}_k$ generated by the elements $\tau_{ij}+\tau_{jl}+\tau_{li}$; then $e_nI=0$. Let $\bar{\mathfrak{A}}=\mathfrak{A}_k/I$. In $\bar{\mathfrak{A}}$, then, the following relations hold:

$$\tau_{12}=\pi_2,\ \tau_{23}=\frac{1}{2}(\pi_3-\pi_2),\ \tau_{34}=\frac{1}{3}(\pi_4-\pi_3),\ \dots,\ \tau_{k-1,k}=\frac{1}{k-1}(\pi_k-\pi_{k-1}).$$

Hence, $\bar{\mathfrak{A}}$ is generated by the π_i. Above we showed that $\pi_i^2=\frac{1}{2}(i-1)(i-2)$ in $\bar{\mathfrak{A}}$. So $\bar{\mathfrak{A}}$ is the Clifford algebra genereated by the images of the π_i for $2\leq i\leq k$.

Further on, $(1-e_k)I=I$; so $\mathfrak{A}_k(1-e_k)\supset I$, but since $\bar{\mathfrak{A}}$ is a Clifford algebra (and, in particular, is simple), then $\mathfrak{A}_k(1-e_k)=I$ and, therefore, $\mathfrak{A}_ke_k\cong\mathfrak{A}_k/(1-e_k)\mathfrak{A}_k\cong\mathfrak{A}_k/I\cong\bar{\mathfrak{A}}$, the Clifford algebra with $k-1$ generators. □

Let λ be a strict partition and t a λ-tableau. For $1\leq i\leq l(\lambda)$ set

$$p_t^i=\sum p_\alpha,\ \text{where } \alpha \text{ runs over the entries of the } i\text{-th row of } t$$

and let σ_i be the element from $\mathfrak{A}_k$ constructed as in Lemma 4.1 from the sequence of numbers that stand in the i-th row.

Let V be an (n,n)-dimensional superspace with $n\geq l(\lambda)$ and $W=V^{\otimes k}$, where $k=\sum\lambda_i$. Let M^t be the subspace of W spanned by the vectors $w\in W$ such that

$$E_{ii}w=\lambda_iw;\quad F_{ii}w=p_t^iw \text{ for } 1\leq i\leq l(\lambda).$$

Set also $\sigma_t=\prod\limits_{1\leq i\leq l(\lambda)}\sigma_i$; let R^t be the set of highest weight vectors that belong to M^t.

4.2. Theorem . a) *As $\mathfrak{A}_k$-module, M^t is isomorphic to $\mathfrak{A}_k\sigma_t$.*

b) *The $\mathfrak{A}_k$-submodule in M^t generated by the $\kappa_{t'}v_t$, where t' has the same rows as t, is an isotypical one and its centralizer is isomorphic to the Clifford algebra $C_{k-l(\lambda)}$.*

Proof. Since $\sigma_tv_t=c\cdot v_t$, heading a) is clear.

The facts $v_t\in M^t$ and $\mathfrak{A}_kv_t\subset R^t$ are also clear.

Let us show first of all that the centralizer of R^t is isomorphic to the Clifford algebra $C_{k-l(\lambda)}$.

Obviously, $\mathfrak{A}_k$ and $C_k\otimes U(\mathfrak{q}(V))$ form a Howe-dual pair in W. By Theorem 3.3.1 the centralizer of the H_k-module R^λ is the Clifford algebra generated by the F_{ii} for $1\leq i\leq l(\lambda)$.

Therefore, the centralizer of the $\mathfrak{A}_k$-module R^λ is the Clifford algebra $C_{k+l(\lambda)}$ generated by the F_{ii} for $1 \leq i \leq l(\lambda)$ and $p_1, \dots, p_k$.

The condition $F_{ii}w = p_t^i w$ is equivalent to another one: $ew = w$ for

$$e = \prod_{1\leq i\leq l(\lambda)} \frac{1}{2}\left(1 - \frac{1}{\lambda_i} p_t^i F_{ii}\right) \text{ and } e^2 = e.$$

Therefore, the centralizer of R^t is isomorphic to $eC_{k+l(\lambda)}e$.

Let $C_{k-l(\lambda)}$ be the subalgebra of C_k generated by the $p_i - p_j$ for i, j that belong to the same row of t. Then it is not difficult to see that $eC_{k+l(\lambda)}e \cong C_{k-l(\lambda)}$ and this proves that the centralizer of R^t is isomorphic to $C_{k-l(\lambda)}$.

Let us prove now that the submodule R_1^t of R^t generated by the elements $\kappa_{t'}v_t$, where t' has, up to permutations, the same rows as t, is isomorphic to R^t. To this end it suffices to prove that any endomorphism of R^t sends R_1^t into itself.

But, indeed, any endomorphism of R^t is a multiplication by $p_i - p_j$ for i, j that belong to the same row of t. Now, it suffices to verify that $(p_i - p_j)\kappa_{t'}v_t \in R_1^t$. But

$$(p_i - p_j)\kappa_{t'}v_t = \sqrt{2}\tau_{ij}s_{ij}\kappa_{t'}v_t = -\sqrt{2}\tau_{ij}\kappa_{s_{ij}(t')}v_{t'}.$$

But the rows of $s_{ij}t'$ consist of the same elements that constitute the rows of t. □

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